Gyrokinetic limit of the 2D Hartree equation in a large magnetic field

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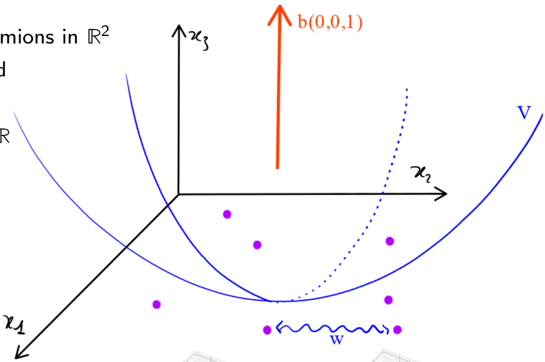
Walkshop in Hagen

13/06/2024

1 Context

Large system of spinless, non relativistic fermions in $\mathbb{R}^2$

- Homogeneous transverse magnetic field
- External potential $V : \mathbb{R}^2 \to \mathbb{R}$
- Radial interaction potential $w : \mathbb{R}^2 \to \mathbb{R}$

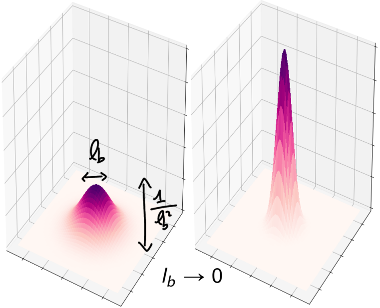


Magnetic length: $l_b := \sqrt{\dfrac{\hbar}{b}}$

$\hbar$: reduced Planck's constant

Semi-classical and high magnetic field limit: $l_b \to 0$

Free ground state density on $\mathbb{R}^2$:



Goal: effective dynamics when $l_b \to 0$

2 Model

Magnetic Laplacian

In a unit system where $m = \frac{1}{2}, c = 1, q = 1$,

$$\mathscr{L}_b \coloneqq (-i\hbar\nabla - bA)^2 \tag{1}$$

Magnetic momentum: $\mathscr{P}_{\hbar,b} \coloneqq -i\hbar\nabla - bA$

Vector potential in symmetric gauge:

$$A \coloneqq \frac{X^\perp}{2} \implies \nabla \wedge A = \begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix} \wedge \begin{pmatrix}A_1\\ A_2\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \tag{2}$$

where $X \coloneqq \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ is the position operator.

Landau level quantization:

$$\mathscr{L}_b \eqqcolon \sum_{n\in\mathbb{N}} 2\hbar b \left(n + \frac{1}{2}\right) \Pi_n \tag{3}$$

Π_n: projection on the n^{th} Landau level

$$\sum_{n\in\mathbb{N}} \Pi_n = \mathrm{Id}_{L^2(\mathbb{R}^2)} \tag{4}$$

Density matrix: $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ such that $\mathrm{Tr}\,(\gamma) = 1, \gamma \geqslant 0$.

Physical density: $\rho_\gamma : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}_+ \\ x & \mapsto & \gamma(x,x) \end{array}$

Hartree equation: Let $\gamma(t)$ be a density matrix $\forall t \in \mathbb{R}_+$,

$$i\hbar\partial_t\gamma = [\mathscr{L}_b + V + w \star \rho_\gamma, \gamma] \tag{5}$$

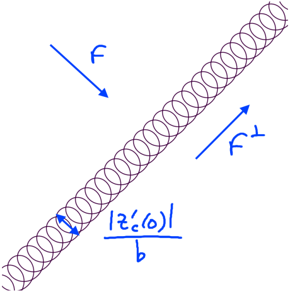
Scaling: $l_b \to 0$ such that $\hbar b = \mathcal{O}(1)$

Classical mechanics: Newton's second law with constant homogeneous force field F

$$Z'' = F + bZ'^{\perp} \tag{6}$$

gives

$$Z(t) = \underbrace{\frac{|Z_c'(0)|}{b}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix}}_{\text{Cyclotron: } Z_c} + \underbrace{\frac{F^{\perp}}{b}t}_{\text{Drift: } Z_d} \tag{7}$$



where we imposed $Z_d(0) = \begin{pmatrix}0\\0\end{pmatrix}$, $\quad Z_c(0) = \frac{|Z_c'(0)|}{b}\begin{pmatrix}1\\0\end{pmatrix}$.

Drift's time scale: $\mathcal{O}(b)$.

Time scaling: $\gamma_b(t) \coloneqq \gamma(bt)$, then

$$il_b^2\partial_t\gamma_b = [\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b] \tag{H}$$

Fermionic Density Matrix (FDM): γ_b density matrix such that $\underbrace{\gamma_b \leqslant 2\pi l_b^2}_{\text{Pauli Principle}}$

Degeneracy per area in a Landau level: $\dfrac{1}{2\pi l_b^2}$

Slater determinants: let $\phi_{1:N} \subset L^2\left(\mathbb{R}^2\right)$ be an orthonormal family, with $N \coloneqq \left\lceil \dfrac{1}{2\pi l_b^2} \right\rceil$, define

$$\psi_N \coloneqq \bigwedge_{i=1}^{N} \phi(i) \tag{8}$$

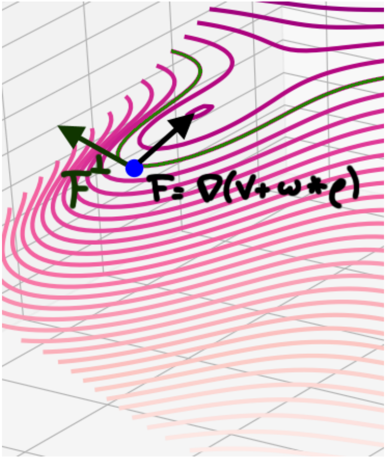
then

$$\gamma_{\psi_N}^{(1)} \coloneqq \frac{1}{N} \sum_{i=1}^{N} |\phi_i\rangle \langle\phi_i| \tag{9}$$

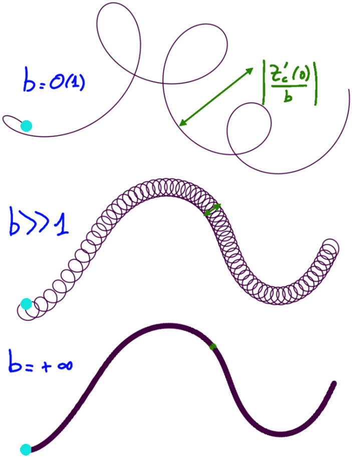
satisfies

$$\mathrm{Tr}\left(\gamma_{\psi_N}^{(1)}\right) = 1 \text{ and } 0 \leqslant \gamma_{\psi_N}^{(1)} \leqslant \frac{1}{N} \leqslant 2\pi l_b^2 \tag{10}$$

Classical dynamics in the potential $V + w \star \rho$ given a density $\rho : \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R}^2 & \to & \mathbb{R}_+ \\ t, z & \mapsto & \rho(t, z) \end{array}$



Level sets of $V + w \star \rho$



Classical trajectories for different b

Drift equation:

$$\partial_t \rho(t, z) + \nabla^\perp (V + w \star \rho(t))(z) \cdot \nabla_z \rho(t, z) = 0 \qquad \text{(D)}$$

3 Main result

$\Gamma(\mu,\nu)$: set of couplings between probabilities $\mu,\nu \in \mathcal{P}\left(\mathbb{R}^2\right)$, 1-Wasserstein metric:

$$W_1(\mu,\nu) \coloneqq \inf_{\pi\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|\, d\pi(x,y) \tag{11}$$

Theorem: *Convergence of densities*

Let γ_b be the solution of (H) given $\gamma_b(0)$ a FDM such that for some $p>7$,

$$\mathrm{Tr}\left(\gamma_b(0)\left(\mathscr{L}_b + V + \frac{1}{2} w * \rho_{\gamma_b}\right)\right) \leqslant C, \quad \mathrm{Tr}\left(\gamma_b(0)\,|X|^p\right) \leqslant C \tag{12}$$

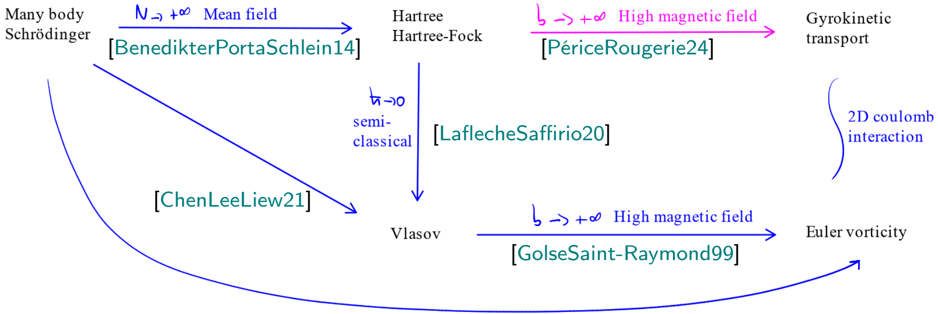
Let ρ solve (D). Assume $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$ and $\nabla w \in L^1\left(\mathbb{R}^2\right), w \in H^2\left(\mathbb{R}^2\right)$.
Then, $\forall t \in \mathbb{R}_+, \forall \varphi \in W^{1,\infty}\left(\mathbb{R}^2\right) \cap H^2\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b}(t) - \rho(t)\right)\right| \leqslant \widetilde{C}\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(W_1\left(\rho_{\gamma_b}(0), \rho(0)\right) + l_b^{\min\left(2\frac{p-7}{4p-7}, \frac{2}{7}\right)}\right) \tag{13}$$

Challenges to overcome:

- Semi-classical phase space: $\mathbb{R}^2 \times \mathbb{N}$
- Controlling the fast cyclotron motion
- Large b/semi-classical time scale

Effective dynamics graph

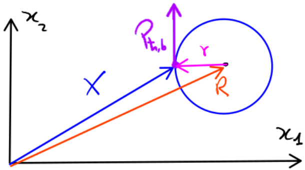


[Porat22], 2D coulomb, $\hbar b \ll \mathcal{O}$ (interactions), phase space $= \mathbb{R}^2 \times \mathbb{R}^2$

Perspectives:

- Reduce regularity assumptions on potentials (methods from [ChongLaflècheSaffirio21])
  - Dynamics of Hushimi functions inside a Landau level
  - Bounds of higher moments of the kinetic energy
- Start from Hartree-Fock and Schrödinger dynamics
- Non homogeneous magnetic field

4 Quantization



Operators:	Position	Annihilation	Creation
Cyclotron	$r := \frac{\mathscr{P}^{\perp}_{\hbar,b}}{b}$	$a_c := \frac{r_2 - ir_1}{\sqrt{2}l_b}$	$a_c^\dagger := \frac{r_2 + ir_1}{\sqrt{2}l_b}$
Drift	$R := X - r$	$a_d := \frac{R_1 - iR_2}{\sqrt{2}l_b}$	$a_d^\dagger := \frac{R_1 + iR_2}{\sqrt{2}l_b}$

Proposition: *Magnetic Laplacian diagonalization*

$\left[a_c, a_c^\dagger\right] = \left[a_d, a_d^\dagger\right] = \mathrm{Id}, [a_c, a_d] = \left[a_c, a_d^\dagger\right] = \left[a_c^\dagger, a_d\right] = \left[a_c^\dagger, a_d^\dagger\right] = 0$, and

$$\varphi_{n,m} := \frac{\left(a_c^\dagger\right)^n \left(a_d^\dagger\right)^m}{\sqrt{n!m!}} \varphi_{0,0} \quad \text{with} \quad \varphi_{0,0}(x) = \frac{1}{\sqrt{2\pi} l_b} e^{\frac{-|x|^2}{4l_b^2}} \tag{14}$$

is a Hilbert basis of $L^2\left(\mathbb{R}^2\right)$ of eigenvectors of $\mathscr{L}_b$. Moreover

$$\Pi_n = \sum_{m \in \mathbb{N}} |\varphi_{n,m}\rangle \langle \varphi_{n,m}|, \quad \mathscr{L}_b = 2\hbar b \left(a_c^\dagger a_c + \frac{1}{2}\right) \tag{15}$$

Coherent state Let $\mathbf{z} \coloneqq z_1 + iz_2 \in \mathbb{C}$, and $z \coloneqq (z_1, z_2) \in \mathbb{R}^2$,

$$\varphi_{n,z} \coloneqq e^{\frac{\bar{\mathbf{z}} a_d^\dagger - \mathbf{z} a_d}{\sqrt{2} l_b}} \varphi_{n,0} = e^{-\frac{|z|^2}{4 l_b^2}} \sum_{m \in \mathbb{N}} \frac{1}{\sqrt{m!}} \left(\frac{\bar{\mathbf{z}}}{\sqrt{2} l_b} \right)^m \varphi_{n,m} \tag{16}$$

then

$$\varphi_{n,z}(x) = \frac{i^n}{\sqrt{2\pi n!} l_b} \left(\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2} l_b} \right)^n e^{-\frac{|x-z|^2 - 2iz^\perp \cdot x}{4 l_b^2}} \tag{17}$$

$$\overline{R} \varphi_{n,z} = \bar{\mathbf{z}} \varphi_{n,z} \tag{18}$$

Phase space projector:

$$\Pi_{n,z} \coloneqq |\varphi_{n,z}\rangle \langle \varphi_{n,z}|, \quad \Pi_z \coloneqq \sum_{n \in \mathbb{N}} |\varphi_{n,z}\rangle \langle \varphi_{n,z}| \tag{19}$$

satisfies

$$\frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \Pi_{n,z} dz = \Pi_n, \quad \Pi_z(x,y) = \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2 - 2i\left(x^\perp \cdot y + 2z^\perp \cdot (x-y)\right)}{4 l_b^2}} \tag{20}$$

so $\nabla_z^\perp \Pi_z(x,y) = \dfrac{i}{l_b^2}(x-y)\Pi_z(x,y)$. In operator form

$$\nabla_z^\perp \Pi_z = \frac{1}{i l_b^2} [\Pi_z, X] \tag{*}$$

5 Semi-classical limit

Let γ_b be a density matrix,

Phase space density $m_{\gamma_b}(n,z) \coloneqq \frac{1}{2\pi l_b^2} \langle \varphi_{n,z} | \gamma_b \varphi_{n,z} \rangle$

Semi-classical density $\rho^{sc}_{\gamma_b}(z) \coloneqq \frac{1}{2\pi l_b^2} \mathrm{Tr}\left(\gamma_b \Pi_z\right)$

Truncated semi-classical density $\rho^{sc,\leqslant M}_{\gamma_b}(z) \coloneqq \sum_{n=0}^{M} m_{\gamma_b}(n,z)$

Proposition: *Convergence of* $\rho^{sc,\leqslant M}_{\gamma_b}$

Let γ_b be a FDM, then $\forall \varphi \in L^\infty \cap H^1\left(\mathbb{R}^2\right)$,

$$\left| \int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho^{sc,\leqslant M}_{\gamma_b} \right) \right| \leqslant C(\varphi)(M^{-\frac{1}{2}} + \underbrace{\sqrt{M} l_b}_{\text{Characteristic length inside NLL}}) \sqrt{\mathrm{Tr}\left(\gamma_b \mathscr{L}_b\right)} \tag{21}$$

We need $1 \ll M \ll \frac{1}{l_b^2}$, higher Landau levels are controlled with the conserved kinetic energy

$$\mathrm{Tr}\left(\gamma_b \mathscr{L}_b\right) = 2\hbar b \sum_{n\in\mathbb{N}} \left(n + \frac{1}{2}\right) \int_{\mathbb{R}^2} m_{\gamma_b}(n,z) dz \tag{22}$$

Proposition: *Gyrokinetic equation for the truncated semi-classical density*

Let $t \in \mathbb{R}_+, \gamma_b(t)$ be a FDM, $W \in W^{4,\infty}(\mathbb{R}^2)$ and assume

$$il_b^2 \partial_t \gamma_b(t) = [\mathscr{L}_b + W, \gamma_b(t)], \quad \mathrm{Tr}\left(\gamma_b(t)\mathscr{L}_b\right) \leqslant C \tag{23}$$

then there exists a choice of $1 \ll M \ll \dfrac{1}{l_b^2}$ such that $\forall \varphi \in L^1 \cap W^{1,\infty}\left(\mathbb{R}^2\right)$,

$$\int_{\mathbb{R}^2} \varphi \left(\partial_t \rho^{sc,\leqslant M}_{\gamma_b(t)} + \nabla^\perp W \cdot \nabla_z \rho^{sc,\leqslant M}_{\gamma_b(t)}\right) \underset{b\to\infty}{\longrightarrow} 0 \tag{24}$$

Convergence $\rho^{sc,\leqslant M}_{\gamma_b} \to \rho$:

- Dobrushin-type stability estimate for the limiting equation
- Use confinement for initial data

6 Central computation

We recall the dynamics and (*)

$$il_b^2 \partial_t \gamma_b = \mathrm{Tr}\left(\mathscr{L}_b + W, \gamma_b\right), \quad \nabla_z^{\perp} \Pi_z = \frac{1}{il_b^2}\left[\Pi_z, X\right]$$

Evolution part

$$\begin{aligned}\partial_t \rho^{sc}_{\gamma_b}(z) =& \frac{1}{2\pi l_b^2}\mathrm{Tr}\left(\Pi_z \partial_t \gamma_b\right) = \frac{1}{2\pi l_b^2}\cdot\frac{1}{il_b^2}\mathrm{Tr}\left(\Pi_z\left[\mathscr{L}_b + W, \gamma_b\right]\right) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma_b\left[\Pi_z, \mathscr{L}_b + W\right]\right)\\ =& \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma_b\left[\Pi_z, W\right]\right)\end{aligned} \tag{25}$$

Spacial part

$$\begin{aligned}\nabla^{\perp} W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) =& -\nabla W(z)\cdot\frac{1}{2\pi l_b^2}\mathrm{Tr}\left(\gamma_b \nabla_z^{\perp}\Pi_z\right) = -\frac{1}{2i\pi l_b^4}\nabla W(z)\cdot\mathrm{Tr}\left(\gamma_b\left[\Pi_z, X\right]\right)\\ =& -\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma_b\left[\Pi_z, \nabla W(z)\cdot X\right]\right)\end{aligned} \tag{26}$$

so

$$\partial_t \rho^{sc}_{\gamma_b}(z) + \nabla^{\perp} W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma_b\left[\Pi_z, W - \nabla W(z)\cdot X\right]\right) \tag{27}$$

where

$$\left[\Pi_z, W - \nabla W(z)\cdot X\right](x,y) = \Pi_z(x,y)\left(W(y) - W(x) - \nabla W(z)\cdot(y-x)\right) \tag{28}$$

Thanks for your attention

References

[PériceRougerie24] D.Périce N.Rougerie. “Gyrokinetic limit of the 2D Hartree equation in a large magnetic field”. In: (2024). DOI: https://arxiv.org/abs/2403.19226.

[Porat22] I.B.Porat. “Derivation of Euler’s equations of perfect fluids from von Neumann’s equation with magnetic field”. In: (2022). DOI: https://doi.org/10.48550/arXiv.2208.01158.

[ChongLaflècheSaffirio21] J.Chong L.Laflèche C.Saffirio. “From Schrödinger to Hartree-Fock and Vlasov equations with Singular potentials”. In: (2021).

[ChenLeeLiew21] M.Liew L.Chen J.Lee. “Combined mean-field and semiclassical limits of large fermionic systems”. In: *J Stat Phys* (2021). DOI: https://doi.org/10.1007/s10955-021-02700-w.

[LaflecheSaffirio20] L.Lafleche C.Saffirio. “Strong semiclassical limit from Hartree and Hartree-Fock to Vlasov-Poisson equation”. In: *arXiv: Mathematical Physics* (2020). DOI: https://arxiv.org/abs/2003.02926.

[BenedikterPortaSchlein14] N.Benedikter M.Porta B.Schlein. “Mean–Field Evolution of Fermionic Systems”. In: *Communications in Mathematical Physics* (2014). DOI: https://arxiv.org/abs/1305.2768.

[GolseSaint-Raymond99] F.Golse L.Saint-Raymond. “The Vlasov–Poisson System with Strong Magnetic Field”. In: *Journal de Mathématiques Pures et Appliquées* (1999). DOI: https://doi.org/10.1016/S0021-7824(99)00021-5.